Analytical density function of sum of Gaussian noise and SαS interference

X.T. Li, L.S. Fan and L.W. Jin

In many practical communication systems, the overall additive noise can be appropriately modelled as the Gaussian noise plus the symmetric α-stable (SαS) distributed interference. A closed-form probability density function of the overall noise is derived to facilitate performance evaluation and system optimisation.

Introduction: Besides Gaussian noise, the impulsive interference modelled as symmetric α-stable (SαS) distribution is often encountered in many practical communication systems. Examples include multiple access systems where impulsive multiple-access interference arises [1] and cognitive networks in which a large amount of spatially Poisson-distributed cognitive radios are simultaneously transmitted [2, 3]. In this Letter, we consider the system affected by the Gaussian noise plus the SαS interference, and derive a closed-form probability density function (PDF) of the overall noise to facilitate performance evaluation and system optimisation. To this end, we first resort to the bi-parameter Cauchy-Gaussian mixture (BCGM) model [4] which provides a concise yet accurate approximation to the SαS PDF. Then we derive a closed-form PDF for the overall noise with $\alpha \in [1, 2]$.

Additive model of overall noise: We consider the communication systems disturbed by both the Gaussian noise and SαS interference. The overall noise is mathematically written as

$$n = n_g + n_\alpha \tag{1}$$

where n_g is the Gaussian noise component with variance of $\gamma_g = \sigma_g^2$ and n_α denotes the SαS interference component. Specifically, n_α can be described by the characteristic function [5]

$$\psi_{n_\alpha}(\theta) = \exp(-\gamma|\theta|^\alpha) \tag{2}$$

where $\alpha \in (0, 2]$ is the characteristic exponent and $\gamma = \sigma^\alpha$ is the dispersion for some $\sigma > 0$. A well-known challenge for SαS is that its PDF is not analytical except for two special cases of Gaussian ($\alpha = 2$) and Cauchy ($\alpha = 1$) distributions.

Analytical PDF of overall noise: From (1), the PDF of n can be computed as

$$f_n(x) = f_{2,\gamma_g}(x) \otimes f_{\alpha,\gamma}(x) \tag{3}$$

where $\otimes$ denotes the convolution operation, and $f_{2,\gamma_g}(x)$ and $f_{\alpha,\gamma}(x)$ represent the PDFs of n_g and n_α, respectively. Since Gaussian density $f_{2,\gamma_g}(x)$ is well known as

$$f_{2,\gamma_g}(x) = \frac{1}{\sqrt{2\pi}\sigma_g}\exp\left(-\frac{x^2}{2\sigma_g^2}\right) \tag{4}$$

we next need to obtain an analytical $f_{\alpha,\gamma}(x)$. From the BCGM model, a very simple yet accurate approximation for the SαS PDF is given by [4],

$$\begin{aligned} f_{\alpha,\gamma}(x) &= (1-\varepsilon)f_{2,\gamma}(x) + \varepsilon f_{1,\gamma}(x) \\ &= \frac{1-\varepsilon}{2\sqrt{\pi}\sigma}\exp\left(-\frac{x^2}{4\sigma^2}\right) + \frac{\varepsilon\sigma}{\pi(x^2+\sigma^2)} \end{aligned} \tag{5}$$

where $\varepsilon \in [0, 1]$ is the mixture ratio, evaluated as

$$\varepsilon = \frac{2\Gamma(-p/\alpha) - \alpha\Gamma(-\rho/2)}{2\alpha\Gamma(-p) - \alpha\Gamma(-\rho/2)} \tag{6}$$

in which $\Gamma(\ )$ denotes the Gamma function and $p = -1/4$ is used. Substituting (4) and (5) into (3) yields the PDF of n as

$$f_n(x) = (1-\varepsilon)f_{2,\gamma_g}(x) \otimes f_{2,\gamma}(x) + \varepsilon f_{2,\gamma_g}(x) \otimes f_{1,\gamma}(x) \tag{7}$$

Since $f_{2,\gamma_g}(x) \otimes f_{2,\gamma}(x)$ represents the density function of the sum of two Gaussian random variables, it can be readily evaluated as

$$f_{2,\gamma_g}(x) \otimes f_{2,\gamma}(x) = \frac{1}{\sqrt{2\pi(\sigma_g^2 + 2\sigma^2)}}\exp\left(-\frac{x^2}{2(\sigma_g^2+2\sigma^2)}\right) \tag{8}$$

Moreover, $f_{2,\gamma_g}(x) \otimes f_{1,\gamma}(x)$ can be calculated as

$$f_{2,\gamma_g}(x) \otimes f_{1,\gamma}(x) = \frac{1}{\pi\sqrt{2\pi}\sigma_g}\int_{-\infty}^{+\infty}\exp\left(-\frac{r^2}{2\sigma_g^2}\right) \times \frac{\sigma}{(x-\tau)^2+\sigma^2}d\tau \tag{9}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_g}\mathrm{Re}\left[w\left(\frac{x}{\sqrt{2}\sigma_g} + j\frac{\sigma}{\sqrt{2}\sigma_g}\right)\right] \tag{10}$$

where [6, equation (7.4.13)] is applied in the last equality, and $w(\ )$ is the Faddeeva function [6] with

$$w(x+jy) = \exp[-(x+jy)^2]\mathrm{erfc}(y-jx) \text{ for real } x \text{ and } y > 0 \tag{11}$$

where erfc( ) is the complex complementary error function [7]. Summarising the results in (8) and (10) finally yields the PDF of the overall noise as

$$\begin{aligned} f_n(x) = &\frac{1-\varepsilon}{\sqrt{2\pi(\sigma_g^2+2\sigma^2)}}\exp\left(-\frac{x^2}{2(\sigma_g^2+2\sigma^2)}\right) \\ &+ \frac{\varepsilon}{\sqrt{2\pi}\sigma_g}\mathrm{Re}\left[w\left(\frac{x}{\sqrt{2}\sigma_g} + j\frac{\sigma}{\sqrt{2}\sigma_g}\right)\right] \end{aligned} \tag{12}$$

Performance evaluation: To verify the proposed studies, we generate two independent sample series of length 10^6: one is SαS series with $\gamma = 1$ and the other is Gaussian series with $\sigma_g = 1$. We then add them to simulate the overall noise. Fig. 1 shows the analytical PDF $f_n(x)$ and the empirical PDF $f_{emp}(x)$, where $\alpha = 1.5$. As observed from Fig. 1, the analytical PDF is close to the actual one, which verifies the derived $f_n(x)$ in (12).

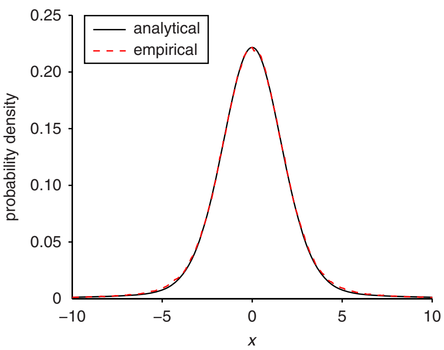

Fig. 1 *Analytical and empirical PDFs with $\alpha = 1.5$*

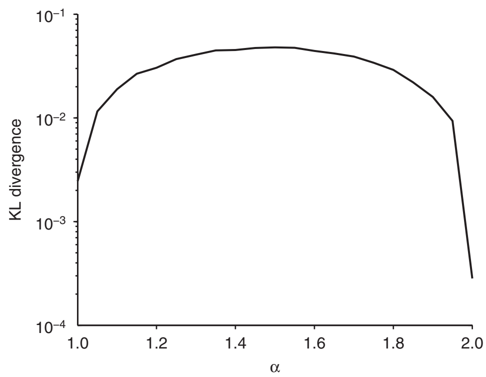

Fig. 2 *KL divergence against α in range of [1, 2]*

We then quantitatively characterise the closeness between the derived analytical PDF $f_n(x)$ and the empirical PDF $f_{emp}(x)$ from the Kullback-Leibler (KL) divergence [8]

$$D_{\mathrm{KL}}(f_n(x)\|f_{emp}(x)) = \int_{-\infty}^{\infty} f_n(x)\log_2\left(\frac{f_x(x)}{f_{emp}(x)}\right)dx \tag{13}$$

To numerically calculate the KL divergence, the integral range in (13) is truncated into [−100, 100], and a step of 0.2 is employed to discretise the integral. The result of KL divergence between the analytical and empirical PDFs is indicated in Fig. 2, where α varies in [1, 2]. In addition, we demonstrate the entropy of the empirical PDF in Fig. 3. From Figs. 2 and 3, we observe that on average, the KL divergence is about 10^{-3} of the associated entropy, which further corroborates the derived analytical PDF of the overall noise.

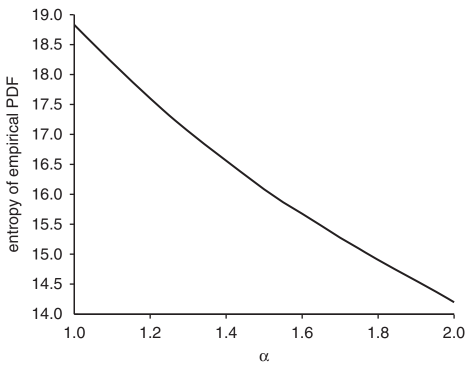

Fig. 3 *Entropy of empirical PDF against* $\alpha \in$ *[1, 2]*

Conclusion: In this Letter, analytical PDFs of the overall noise have been derived for the communication systems suffering from both Gaussian noise and SαS-distributed interference. Numerical simulations have illustrated the accuracy of the proposed analytical PDF. The authors feel that the results presented in this Letter may help to alleviate the simulation burdens and facilitate the optimisation for similar communication systems.

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One or more of the Figures in this Letter are available in colour online.

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